LESSON

1 Functions

Calculus is the study of change. It is often important to know when something is increasing, when it is decreasing, and when it hits a high or low point. Much of the business of finance depends on predicting the high and low points for prices. In science and engineering, it is often essential to know precisely how fast quantities such as temperature, size, and speed are changing. Calculus is the primary tool for calculating such changes.

Numbers, which are the focus of arithmetic, are no longer the objects of our study. This is because they do not change. The number 5 will always be 5. It never goes up or down. Thus, we need to introduce a new sort of mathematical object, something that *can* change. These objects, the centerpiece of calculus, are functions.

Functions

A *function* is a way of matching up one set of numbers with another. The first set of numbers is called the *domain.* For each of these numbers in a set, the function assigns exactly one number from the other set, the *range.*

Parentheses Hint

It is true that in algebra, everyone is taught "parentheses mean multiplication." This means that $5(2 + 7) = 5(9) = 45$. If x is a variable, then $x(2 + 7) = x(9) = 9x$. However, if f is the name of a function, then $f(2 + 7) = f(9) =$ the number to which f takes 9. The expression $f(x)$ is pronounced "f of x" and not "f times x." This can be confusing, so an apology is necessary. Mathematicians use parentheses to mean several different things and expect everyone to know the difference. Sorry!

For example, the domain of the function could be the numbers 1, 4, 9, 25, and 100; and the range could be 1, 2, 3, 5, and 10. Suppose the function takes 1 to 1, 4 to 2, 9 to 3, 25 to 5, and 100 to 10. This could be illustrated by the following:

$$\begin{aligned} 1 &\rightarrow 1 \\ 4 &\rightarrow 2 \\ 9 &\rightarrow 3 \\ 25 &\rightarrow 5 \\ 100 &\rightarrow 10 \end{aligned}$$

Because we sometimes use several functions at the same time, we give them names. Let us call the function we just mentioned by the name *Eugene.* Thus, we can ask, "Hey, what does Eugene do with the number 4?" The answer is "Eugene takes 4 to the number 2."

Mathematicians are notoriously lazy, so we try to do as little writing as possible. Thus, instead of writing "Eugene takes 4 to the number 2," we often write "Eugene(4) = 2" to mean the same thing. Similarly, we like to use names that are as short as possible, such as f (for function), g (for function when f is already being used), h, and so on. The trigonometric functions in Lesson 4 all have three-letter names like sin and cos, but even these are abbreviations. So let us save space and use f instead of Eugene.

Because the domain is small, it is easy to write out everything:

$$\begin{aligned} f(1) &= 1 \\ f(4) &= 2 \\ f(9) &= 3 \\ f(25) &= 5 \\ f(100) &= 10 \end{aligned}$$

However, if the domain were large, this would get very tedious. It is much easier to find a pattern and use that pattern to describe the function. Our function f just happens to take each number of its domain to the square root of that number. Therefore, we can describe f by saying:

f(a number) = the square root of that number

Of course, anyone with experience in algebra knows that writing "a number" over and over is a waste of time. Why not just pick a *variable* to represent the number? Just as f is our favorite name for functions, little x is the most beloved of all variable names. Here is the way to represent our function f with the absolute least amount of writing necessary:

$$f(x) = \sqrt{x}$$

This tells us that putting a number into the function f is the same as putting it into $\sqrt{}$. Thus,

$f(25) = \sqrt{25} = 5$ and $f(4) = \sqrt{4} = 2$.

Example

Find the value of $g(3)$ if $g(x) = x^2 + 2$.

Solution

Replace each occurrence of x with 3.

$g(3) = 3^2 + 2$

Simplify.

$g(3) = 9 + 2 = 11$

Example

Find the value of $h(-2)$ if $h(t) = t^3 - 2t^2 + 5$.

Solution

Replace each occurrence of t with -2.

$h(-2) = (-2)^3 - 2(-2)^2 + 5$

Simplify.

$h(-2) = -8 - 2(4) + 5 = -8 - 8 + 5 = -11$

▶ Practice

1. Find the value of $f(5)$ when $f(x) = 2x - 1$.
2. Find the value of $g(-3)$ when $g(x) = x^3 + x^2 + x + 1$.
3. Find the value of $h\left(\frac{1}{2}\right)$ when $h(t) = t^2 + \frac{3}{4}$.
4. Find the value of $f(7)$ when $f(x) = 2$.
5. Find the value of $k(4)$ when $k(u) = u^2 + 2u - \frac{12}{u}$.
6. Find the value of $h(64)$ when $h(x) = \sqrt{x} - \sqrt[3]{x}$.
7. Suppose that after t seconds, a rock thrown off a bridge has height $s(t) = -16t^2 + 20t + 100$ feet off the ground. How high is it after 3 seconds?
8. Suppose that the profit on making and selling x cookies is $P(x) = \frac{x}{2} - \frac{x^2}{10{,}000} - \$10$. How much profit is made on 100 cookies?

▶ Plugging Variables into Functions

Variables can be plugged into functions just as easily as numbers can. Often, though, they can't be simplified as much.

Example

Simplify $f(w)$ if $f(x) = \sqrt{x} + 2x^2 + 2$.

Solution

Replace each occurrence of x with w.

$f(w) = \sqrt{w} + 2w^2 + 2$

That is all we can say without knowing more about w.

Example

Simplify $g(a + 5)$ if $g(t) = t^2 - 3t + 1$.

Solution

Replace each occurrence of t with $(a + 5)$.

$g(a + 5) = (a + 5)^2 - 3(a + 5) + 1$

Multiply out $(a + 5)^2$ and $-3(a + 5)$.

$g(a + 5) = a^2 + 10a + 25 - 3a - 15 + 1$

Simplify.

$g(a + 5) = a^2 + 7a + 11$

Example

Simplify $\dfrac{f(x+a)-f(x)}{a}$ if $f(x) = x^2$.

Solution

Start with what needs to be simplified.

$$\frac{f(x+a)-f(x)}{a}$$

Use $f(x) = x^2$ to evaluate $f(x+a)$ and $f(x)$.

$$\frac{(x+a)^2 - x^2}{a}$$

Multiply out $(x+a)^2$.

$$\frac{x^2 + 2xa + a^2 - x^2}{a}$$

Cancel out the x^2 and the $-x^2$.

$$\frac{2xa + a^2}{a}$$

Factor out an a.

$$\frac{(2x+a)a}{a}$$

Cancel an a from the top and bottom.

$2x + a$

▶ Practice

Simplify the following.

9. $f(y)$ when $f(x) = x^2 + 3x - 1$

10. $f(y+1)$ when $f(x) = x^2 + 3x - 1$

11. $f(x+a)$ when $f(x) = x^2 + 3x - 1$

12. $g(x^2 + \sqrt{x})$ when $g(t) = \frac{8}{t} - 6t$

13. $g(2x) - g(x)$ when $g(t) = \frac{8}{t} - 6t$

14. $f(x+a) - f(x)$ when $f(x) = x^2 + 4x - 5$

15. $\dfrac{h(x+a) - h(x)}{a}$ when $h(x) = 3x + 2$

16. $\dfrac{g(x+a) - g(x)}{a}$ when $g(x) = x^2 - 2x + 1$

▶ Composition

Now that we can plug anything into functions, we can plug one function into another. This is called *composition.* The composition of function f with function g is written $f \circ g$. This means to plug g into f like this:

$f \circ g(x) = f(g(x))$

It may seem that f comes first in $f \circ g(x)$, reading from left to right, but actually, the g is closer to the x. This means that the function g acts on the x first.

Example

If $f(x) = \sqrt{x} + 2x$ and $g(x) = 4x + 7$, then what is the composition $f \circ g(x)$?

Solution

Start with the definition of composition.

$f \circ g(x) = f(g(x))$

Use $g(x) = 4x + 7$.

$f \circ g(x) = f(4x + 7)$

Replace each occurrence of x in f with $4x + 7$.

$f \circ g(x) = \sqrt{4x+7} + 2(4x + 7)$

Simplify.

$f \circ g(x) = \sqrt{4x+7} + 8x + 14$

Conversely, to evaluate $g \circ f(x)$, we compute:

$g \circ f(x) = g(f(x))$

Use $f(x) = \sqrt{x} + 2x$.

$g \circ f(x) = g(\sqrt{x} + 2x)$

Replace each occurrence of x in g with $\sqrt{x} + 2x$.

$g \circ f(x) = 4(\sqrt{x} + 2x) + 7$

Simplify.

$g \circ f(x) = 4\sqrt{x} + 8x + 7$

Notice that $f \circ g(x)$ and $g \circ f(x)$ are different. This is usually the case.

Example

If $f(x) = x^2 + 2x + 1$ and $g(x) = 5x + 1$, then what is $f \circ g(x)$?

Solution

Start with the definition of composition.

$f \circ g(x) = f(g(x))$

Use $g(x) = 5x + 1$.

$f \circ g(x) = f(5x + 1)$

Replace each occurrence of x in f with $5x + 1$.

$f \circ g(x) = (5x + 1)^2 + 2(5x + 1) + 1$

Simplify.

$f \circ g(x) = 25x^2 + 20x + 4$

▶ Practice

Using $f(x) = \frac{1}{x}$, $g(x) = x^3 - 2x^2 + 1$, and $h(x) = x - \sqrt{x}$, simplify the following compositions.

17. $f \circ g(x)$

18. $g \circ f(x)$

19. $f \circ h(t)$

20. $f \circ f(x)$

21. $h \circ h(x)$

22. $g \circ h(9)$

23. $h \circ f \circ g(x)$

24. $f \circ h \circ f(2x)$

▶ Domains

In the beginning of the lesson, we defined the function Eugene as:

$f(x) = \sqrt{x}$

However, we left out a crucial piece of information: the domain. The domain of this function consisted of only the numbers 1, 4, 9, 25, and 100. Thus, we should have written

$f(x) = \sqrt{x}$ if $x = 1, 4, 9, 25,$ or 100

Usually, the domain of a function is not given explicitly like this. In such situations, it is assumed that the domain is as large as it possibly can be. The domain

consists of all numbers that don't violate one of the following two fundamental prohibitions:

- Never divide by zero.
- Never take an even root of a negative number.

If you divide by zero, the entire numerical universe will collapse down to a single point. If dividing by zero were allowed, then all numbers would be equal. Four would equal five. Negative and positive would be equivalent. "It's all the same to me" would be the correct answer to every math question. While this might be appealing to some people, it would make calculus, the study of change, impossible. If only one number existed, there could be no change. Thus, we automatically rule out any situation where division by zero might occur.

Example

What is the domain of $f(x) = \dfrac{3}{x-2}$?

Solution

We must never let the denominator $x - 2$ be zero, so x cannot be 2. Therefore, the domain of this function consists of all real numbers except 2.

The prohibition against even roots (like square roots) of negative numbers is less severe. An even root of a negative number is an imaginary number. Useful mathematics can be done with imaginary numbers. However, for the sake of simplicity, we will avoid them in this book.

Example

What is the domain of $g(x) = \sqrt{3x+2}$?

Solution

The numbers in the square root must not be negative, so $3x + 2 \geq 0$, thus $x \geq -\frac{2}{3}$. The domain consists of all numbers greater than or equal to $-\frac{2}{3}$.

Do note that it is perfectly okay to take the square root of zero, since $\sqrt{0} = 0$. It is only when numbers are less than zero that even roots become imaginary.

Example

Find the domain of $k(x) = \dfrac{\sqrt{4-x}}{x^2+5x+6}$.

Solution

To avoid dividing by zero, we need $x^2 + 5x + 6 \neq 0$, so $(x+3)(x+2) \neq 0$, thus $x \neq -3$ and $x \neq -2$. To avoid an even root of a negative number, $4 - x \geq 0$, so $x \leq 4$. Thus, the domain of k is $x \leq 4$, $x \neq -3$, $x \neq -2$.

► Practice

Find the domain of each of the following functions.

25. $f(x) = \dfrac{1}{(x+3)(x-5)}$

26. $h(x) = \sqrt{x+1}$

27. $k(t) = \dfrac{1}{\sqrt{t+5}}$

28. $g(x) = x^2 + 5x - 6$

29. $f(a) = \dfrac{3x+7}{a}$

30. $h(x) = \sqrt[3]{x}$

31. $k(x) = \dfrac{\sqrt[4]{2-x}}{x+8}$

32. $g(u) = \dfrac{8u}{\sqrt{4+3u(u+3)}}$